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REINFORCED CONCRETE SKEWED RIGID-FRAME AND ARCH BRIDGES

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PAPERS

REINFORCED CONCRETE SKEWED RIGID-FRAME AND ARCH BRIDGES

By Maurice Barron,[1] M. ASCE

Synopsis

This paper presents a method of analysis and design by which the effects of skew on a barrel arch or rigid-frame bridge may be evaluated separately. The skew problem is thus resolved into a sort of secondary stress analysis in which the ordinary stresses for a rectangular structure are used as primary stresses. The procedure is extended so that final steel areas of reinforcement and unit stresses are determined as functions of the areas and unit stresses for the rectangular structure. Equations and transformations are derived for all applied loads and for volume changes. Criteria are presented for the effects of skew. For single-span or double-span structures with less then critical skew, no skew analysis is required. For structures with greater than critical skew, an independent adjustment is shown for analysis and design.

The practical significance of this development can better be appreciated if one considers the modern parkway or expressway as a sinuous ribbon of constant width. If the horizontal clearances are constant, every overpass on the parkway has the same square span. The single-span or the double-span structures may be so proportioned that they differ (structurally) only in width or skew, and sometimes in the depth of foundations. Therefore, one basic rectangular structure becomes the parent of a family of bridges.

Another feature of practical and theoretical significance is the fact that, for the rectangular structure, a designer may use any preferred method of analysis and design. If required, the independent secondary analysis for skew effects may be readily superimposed.

The results obtained by this method have been compared with the results obtained by the analysis of existing structures designed by current methods which involve the numerical solution of several simultaneous equations. The agreement is convincing evidence of the accuracy of the proposed methods.

Note.—Written comments are invited for publication; the last discussion should be submitted by September 1, 1950.

[1] Farkas & Barron, Cons. Engrs., New York, N. Y.

Introduction

Modern express highways and channelization of traffic have created problems in the economic planning and execution of highway projects. The high speed through highway includes a minimum of three lanes in each direction, and safety considerations have brought about the center mall separating opposing traffic. Thus, the typical cross section will have a reasonable shoulder on each side and two multiple-lane roadways separated by a center mall. The highway will cross intersecting highways usually at some skew, the right-angle intersection being a rare exception. Therefore, the structure separating the grades will usually be a skewed bridge. Maladjustment of alinement to avoid skewed structures is not tolerated in current practice. The double-span rigid-frame bridge has become a recognized type of structure admirably suited to the imposed conditions, the center mall being used to accommodate the center pier.

A single-span structure is used where no center mall is provided, at traffic interchanges where center piers are undesirable, or where opposing traffic is widely divided and two separate structures are indicated.

In 1924, J. Charles Rathbun, M. ASCE, presented a pioneering paper[2] on the theory of the skewed arch which demonstrated the derivation of the elastic equations for the fixed skewed arch ring. Later, at the request of Arthur G. Hayden, M. ASCE, Mr. Rathbun extended his theory to include the elastic equations for the hinged concrete rigid-frame bridge.

This extension was revised, adapted to office use, and first published by Mr. Hayden in 1931.[3] The numerical work required by this method is so difficult, even for single-span structures, that the skewed frame or arch was often abandoned in favor of more easily designed types. A worthy step toward a practical simplification for single-span skewed structures was incorporated in a paper[4] by Richard M. Hodges, in 1943.

The current theory is so complex and the amount of computation required for double-span skewed structures is so much greater than for single-span structures that the former are being used less and less. This is most unfortunate because the need for double-span structures is increasing as more parkways and freeways are built or projected.

Notation.—The letter symbols introduced in this paper are defined where they first appear, in the text or in the illustrations, and are assembled for reference alphabetically in the Appendix.

Derivation of Elastic Equations; Single Span

The conventional structural analysis of hinged single-span skew arches and rigid-frame bridges assumes a rectangular coordinate space system with a reaction force applied in the direction of each of the three mutually perpendicular axes, and a reaction moment about each of two of these three axes. The removal of the redundant reaction forces and redundant reaction moments

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[2] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 611.

[3] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 1931.

[4] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 913.

leaves a statically determinate system. This system is usually called the base frame, transformed structure, or residual structure. For each redundant, an elastic equation may be written so that, simultaneously, the redundants satisfy the elastic requirements for the residual frame, and the remaining reactions independently satisfy the laws of statics.

Any structural system that is statically indeterminate to the nth degree will have n redundants and the simultaneous solution of n elastic equations is required to determine the redundants. If the n redundants are removed and the structure is loaded, elastic deformations in the direction opposite to each redundant will occur. The redundants must produce equal and opposite deformations simultaneously in each of the n directions. When n is equal to 4, this condition may be written mathematically as follows:

$$\left.\begin{aligned} \delta_a &= 0 = A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} + \Delta_a \\ \delta_b &= 0 = A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} + \Delta_b \\ \delta_c &= 0 = A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} + \Delta_c \\ \delta_d &= 0 = A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} + \Delta_d \end{aligned}\right\}\quad \ldots\ldots\ldots\ldots(1)$$

By transposition,

$$\left.\begin{aligned} A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} &= -\Delta_a \\ A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} &= -\Delta_b \\ A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} &= -\Delta_c \\ A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} &= -\Delta_d \end{aligned}\right\}\quad \ldots\ldots\ldots\ldots(2)$$

In Eqs. 1 and 2: Symbols A, B, C, and D are the four redundants for a single-span, hinged skewed structure as shown in Fig. 1; δ_{aa}, δ_{bc}, δ_{ad}, etc., denote the deformation at the point of application, and in the direction, of the redundant indicated by the first subscript, due to a unit load (or unit moment) acting at the point of application, and in the direction, of the redundant indicated by the second subscript; and Δ_a, Δ_b, Δ_c, and Δ_d denote the deformation of the residual structure at the point of application, and in the direction, of the redundants A, B, C, and D, due to loads, volume changes, or movements of the foundations.

Fig. 1 shows a plan and square section of the residual structure for an elliptical skewed arch. The right support is assumed to be sustained by a frictionless ball and socket on frictionless rollers. A rigid (inelastic) bracket is attached to the residual structure above the ball and socket, and the bracket extends to the elastic center which is the point of application of the redundant B. Each redundant is positive in the direction shown. The redundant A is applied to the rigid bracket as shown in plan and section of Fig. 1. The redundant D is applied about the x-axis. The redundant C is applied about the y-axis, and the redundant B is applied to the bracket at the elastic center on the y-axis.

The left support of the residual structure is assumed to be a shaft that permits frictionless rotation about the z-axis. Section LL, Fig. 1, shows a cut section at point P. There are three mutually perpendicular axes, u, v, and z, at the cut section.

The moment stresses at the cut section are shown as M_z, M_v, and M_u, which are the moments about each of the mutually perpendicular axes at point P. There is also a normal thrust T_v, a cross shear T_z, and a radial shear T_u. The moment about the u-axes, M_u, is indicated by a dotted ellipse. Although this

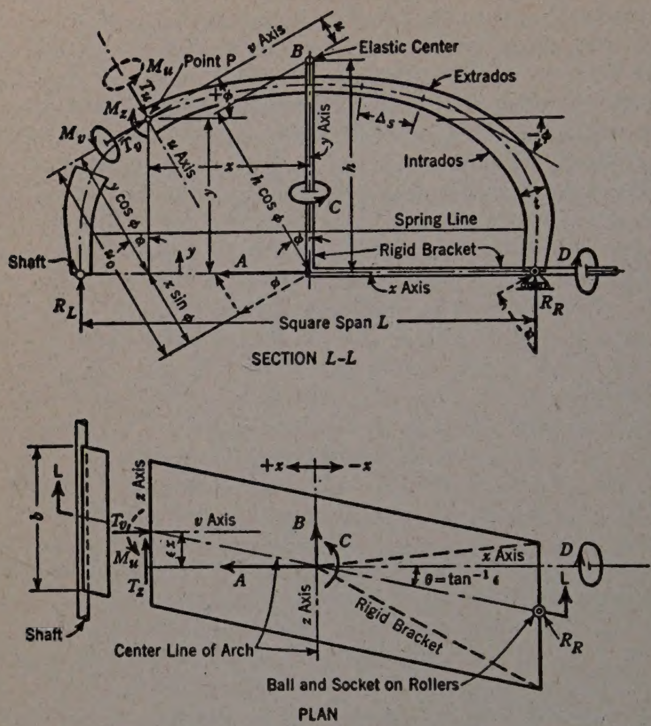

FIG. 1

moment is quite large in value, the section over which it acts is the full width of the bridge, and the resulting stresses are very small. Therefore, the moment M_u is neglected. As customary, the radial shear T_u is also neglected, since it results in negligible stresses. Table 1 shows the unit moments m at any section of the residual structure for each unit redundant.

TABLE 1.—MOMENTS m FOR UNIT REDUNDANTS

Redundant	m_z	m_v
$A = +1$	$-y$	$+\epsilon x \sin \phi$
$B = +1$	0	$-u$
$C = +1$	0	$-\sin \phi$
$D = +1$	0	$+\cos \phi$

A complete understanding of the basic relations shown in Fig. 1 will remove the principal source of difficulty in the analysis of skewed structures. If the designer masters these basic relations and can derive the values shown in Table 1, he has mastered the major part of all that has been written on the analysis of this type of skewed structure.

The following equations are readily derived from Fig. 1 and from Table 1:

$$u_o = x \sin \phi + y \cos \phi \quad \text{(3a)}$$

$$u = u_o - h \cos \phi \quad \text{(3b)}$$

$$M_z = M_{oz} - A\, y \quad \text{(4a)}$$

$$M_v = M_{ov} + \epsilon\, A\, x \sin \phi - B\, u - C \sin \phi + D \cos \phi \quad \text{(4b)}$$

$$T_v = V \sin \phi + A \cos \phi \quad \text{(5a)}$$

and

$$T_z = B \quad \text{(5b)}$$

Since the single-span hinged skewed arch or rigid-frame bridge is a structure which is indeterminate in the fourth degree, Eqs. 1 and 2 will apply. The right-hand term in each Eq. 2 represents the deformation of the residual structure in the direction of each redundant because of the loads on the structure. These deformations may also be the result of a volume change (temperature change, rib shortening from axial stress, plastic yield, etc.), or from yielding of the foundations.

Unit Deflections.—Each term on the left side of Eqs. 2 consists of two parts, one a redundant and the other a unit deformation. The redundant cannot be made zero, but an analysis of each unit deformation will disclose the possibility of making it zero. In Eqs. 2, the unit deformations become coefficients for the redundants.

The equation for each unit deformation δ may be written directly from Table 1 using the Maxwell theorem of deflection. This theorem applies to both the rectangular elastic system, and to the torsional elastic system. The basic formulas are as follows:

For the rectangular elastic system—

$$\Delta_R = \int \frac{M_z\, m_z\, ds}{E_c\, I} = \sum \frac{M_z\, m_z\, \Delta s}{E_c\, I} \quad \text{(6a)}$$

for the torsional elastic system—

$$\Delta_T = \int \frac{M_v\, m_v\, ds}{E_s\, F} = \sum \frac{M_v\, m_v\, \Delta s}{E_s\, F} \quad \text{(6b)}$$

In Eqs. 6, M_z and M_v are the components of the moment acting on the surface cut by a radial plane through point P about the z-axis and the v-axis, respectively. Moment M_v is positive when it causes the obtuse angle of the section on which it is acting to deflect toward the center of curvature, and M_z is positive when it produces compression in the extrados fibers. Symbols m_z and m_v are the moments corresponding to M_z and M_v when a unit load is applied at a point m. The factor F can be computed[5] by the equation:

$$\frac{1}{F} = \frac{205\,(t^2 + b^2)}{180\, b^3\, t^3}\, \pi \quad \text{(7a)}$$

[5] *Bulletin No. 3*, Faculty of Applied Science and Eng., School of Eng. Research, Univ. of Toronto, Toronto, Ont., Canada.

which, in turn, may be written

$$\frac{1}{F} = \frac{205}{180}\pi\left(\frac{1}{b^3 t} + \frac{1}{b\,t^3}\right) \qquad \text{(7b)}$$

Since b is usually large compared to t, the term $1/(b^3 t)$ may be neglected; therefore,

$$\frac{1}{F} = \frac{205\,\pi}{180\,b\,t^3} = \frac{3.578}{b\,t^3} = G \qquad \text{(7c)}$$

Also

$$\frac{1}{I} = \frac{12}{b\,t^3} = G_1 \qquad \text{(7d)}$$

Eq. 6b may be written

$$\Delta_T = r\int\frac{M_v\,m_v\,ds}{E_c\,I} = r\sum\frac{M_v\,m_v\,\Delta s}{E_c\,I} \qquad \text{(8)}$$

in which r is an elastic element defined as $\frac{E_c\,I}{E_s\,F} = \lambda\,\frac{3.578}{12} = 0.685.$

Table 2 results from substituting the values for Table 1 into Eqs. 6. It shows the formula used for the evaluation of the unit deformations, and also the locations of the sections that contribute the major part to the unit deforma-

TABLE 2.—UNIT DEFLECTIONS

Deflection (1)	M_v (2)	m (3)	Equation[a] (4)	Major contribution by sections at the: (5)	Remarks (6)
(a) UNIT DEFORMATION FOR THE RECTANGULAR ELASTIC EQUATION (EQ. 6a)					
$\delta_{aa,R}$	$-y$	$-y$	$\Sigma(y^2/I)$	Crown	....
(b) UNIT DEFORMATION FOR THE TORSIONAL ELASTIC SYSTEMS (EQ. 6b)					
$\delta_{aa,T}$	$+\epsilon x \sin\phi$	$+\epsilon x \sin\phi$	$+\epsilon^2 r\,\Sigma(x^2\sin^2\phi/I)$	Knee	....
$\delta_{bb,T}$	$-u$	$-u$	$+r\,\Sigma(u^2/I)$	Knee	....
$\delta_{cc,T}$	$-\sin\phi$	$-\sin\phi$	$+r\,\Sigma(\sin^2\phi/I)$	Knee	....
$\delta_{dd,T}$	$+\cos\phi$	$+\cos\phi$	$+r\,\Sigma(\cos^2\phi/I)$	Crown	....
$\delta_{ab,T}$	$+\epsilon x \sin\phi$	$-u$	$-\epsilon r\,\Sigma(x\,u\sin\phi/I)$	Knee	$\delta_{ab} = \delta_{ba}$
$\delta_{ad,T}$	$+\epsilon x \sin\phi$	$+\cos\phi$	$+\epsilon r\,\Sigma(x\sin\phi\cos\phi/I)$	Knee to quarter point	$\delta_{ad} = \delta_{da}$
$\delta_{ac,T}$	$+\epsilon x \sin\phi$	$-\sin\phi$	$-\epsilon r\,\Sigma(x\sin^2\phi/I)$	....	$\delta_{ac} = \delta_{ca} = 0$
$\delta_{bc,T}$	$-u$	$-\sin\phi$	$+r\,\Sigma(u\sin\phi/I)$	....	$\delta_{bc} = \delta_{cb} = 0$
$\delta_{dc,T}$	$+\cos\phi$	$-\sin\phi$	$-r\,\Sigma(\sin\phi\cos\phi/I)$	....	$\delta_{dc} = \delta_{cd} = 0$
$\delta_{bd,T}$	$-u$	$+\cos\phi$	$-r\,\Sigma(u\cos\phi/I)$	....	$\delta_{bd} = \delta_{db} = 0$

[a] The equations in Col. 4, multiplied by $\Delta s/E_c$, result from substituting values from Table 1 in Eqs. 6.

tion. It should be noted that four unit deformations have similar double subscripts—δ_{aa}, δ_{bb}, δ_{cc}, δ_{dd}. The unit deformations with similar double subscripts are the sum of squares and always positive, and they are of major

importance. Deformation δ_{aa} is the only unit deformation to which both the rectangular and torsional elastic systems contribute a part, as shown in Table 2, and subsequently in Eq. 13*a*. The other unit deformations have dissimilar subscripts.

Two unit deformations with dissimilar subscripts, which differ only in that there is an inversion in the subscripts, are equal in accordance with Maxwell's theorem of reciprocal deflections. This statement is expressed mathematically as $\delta_{mn} = \delta_{nm}$ and several applications are shown in Table 2. Also, as shown in Table 2, eight of the unit deformations may be equated to zero; δ_{ac} $(= \delta_{ca})$, δ_{bc} $(= \delta_{cb})$, and δ_{dc} $(= \delta_{cd})$ are equal to 0 by symmetry. The last unit deformation in Table 2, deformation δ_{bd} $(= \delta_{db})$, is made equal to 0 by the proper location of the elastic center. This location is established by substituting Eqs. 3*b* into the equation for δ_{bd} (last item of Col. 4, Table 2), equating to zero, and solving for h; thus: $\delta_{bd} = 0 = \sum \frac{u \cos\phi\, \Delta s}{E_s F} = r \sum \frac{(u_o - h\cos\phi)\cos\phi\, \Delta s}{E_c I}$ $= r \sum \frac{u_o \cos\phi\, \Delta s}{E_c I} - h\, r \sum \frac{\cos^2\phi\, \Delta s}{E_c I} = 0$. Therefore,

$$h = \frac{\Sigma(u_o \cos\phi/I)}{\Sigma(\cos^2\phi/I)} \qquad (9)$$

This elastic center always occurs a little above the axis of the structure and on the center line of symmetrical structures as shown in Fig. 1.

The elastic center, as used in this demonstration, is a particular point in space which is the end of a rigid bracket of infinite stiffness. This bracket is attached to the structure so that, by an appropriate transformation, the effects of stresses, which would normally result at the point of attachment, are exactly duplicated by another set of stresses at the elastic center. This second set of stresses is much more easily determined than the set at the point of attachment. Applying the redundant B at the elastic center, the deformation $B\,\delta_{bd} = 0$, which means that the angular rotation in the direction of the redundant D is equal to zero for all values of B. Similarly, $D\,\delta_{db} = 0$, which means that the deformation in the direction of the redundant B is equal to zero for all values of the redundant D.

Solution of Elastic Equations; Single Span

Rewriting Eqs. 2 and dropping the zero terms as shown in Table 2,

$$A\,\delta_{aa} + B\,\delta_{ab} \qquad + D\,\delta_{ad} = -\Delta_a \qquad (10a)$$

$$A\,\delta_{ab} + B\,\delta_{bb} \qquad = -\Delta_b \qquad (10b)$$

$$C\,\delta_{cc} \qquad = -\Delta_c \qquad (10c)$$

$$A\,\delta_{ad} \qquad + D\,\delta_{dd} = -\Delta_d \qquad (10d)$$

From Eq. 10*b*,

$$B = -\frac{\delta_{ab}}{\delta_{bb}} A - \frac{1}{\delta_{bb}} \Delta_b \qquad (11b)$$

From Eq. 10*c*,

$$C = -\frac{1}{\delta_{cc}}\Delta_c \quad \text{(11c)}$$

From Eq. 10*d*,

$$D = -\frac{\delta_{ad}}{\delta_{dd}}A - \frac{1}{\delta_{dd}}\Delta_d \quad \text{(11d)}$$

Substituting Eqs. 11 into Eq. 10*a*,

$$A\left(\delta_{aa} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}}\right) = -\Delta_a + \frac{\delta_{ab}}{\delta_{bb}}\Delta_b + \frac{\delta_{ad}}{\delta_{dd}}\Delta_d \quad \text{(12)}$$

From Tables 1 and 2,

$$\delta_{aa} = \delta_{aa,R} + \delta_{aa,T} \quad \text{(13a)}$$

and

$$\Delta_a = \Delta_{a,R} + \Delta_{a,T} \quad \text{(13b)}$$

Let

$$\delta_{aa,T} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}} = Q_1 \quad \text{(14a)}$$

$$\frac{\delta_{ab}}{\delta_{bb}} = Q_2 \quad \text{(14b)}$$

$$\frac{\delta_{ad}}{\epsilon\,\delta_{dd}} = Q_3 \quad \text{(14c)}$$

$$\frac{1}{\delta_{aa,R}} = Q_4 \quad \text{(14d)}$$

$$\frac{1}{\delta_{bb}} = Q_5 \quad \text{(14e)}$$

$$\frac{1}{\delta_{cc}} = Q_6 \quad \text{(14f)}$$

and

$$\frac{1}{\delta_{dd}} = Q_7 \quad \text{(14g)}$$

At this point, where various factors enter into the solution, the order of a quantity should be mentioned and some explanation given for what is meant by order. In many of their derivations, mathematicians drop differentials of higher order without diminishing the value of the demonstration. Similarly, structural engineers neglect terms where their effect on the final answer is small and of no practical importance. Thus, in the usual analysis of right arches, and rigid frames, the deflections caused by shear, axial stress, and plastic flow are usually of a different order from the deflections produced by primary flexure; and therefore they are ordinarily neglected. If these factors of little importance have different algebraic signs, they are compensating and the order of their total may then change to one of less importance.

This analysis consists primarily of combining terms in a particular order so that certain terms and combinations of terms tend to cancel each other. The

parts that do not cancel are easily determined as secondary effects but they are usually of an unimportant nature. Thus, it is shown that the effects of skew may be found directly from the stresses determined for the corresponding rectangular structure. In effect, the latter part of the torsional stress analysis, therefore, is a kind of secondary stress problem.

It has been found, as for example in Eq. 12, that the combination of terms which comprises Q_1 is of very little importance when compared with the value of $\delta_{aa,R}$. In the usual rigid frame with a skew of 50° the ratio of Q_1 to $\delta_{aa,R}$ is of the order 1:2,000. In Eq. 12 it is proper, therefore, to drop out Q_1 inasmuch as it will affect the coefficient of A by only 0.05%. Similarly, for rigid frames, Q_2 has been found to equal almost exactly $-\epsilon$. Even in the full centered arch the error in assuming that $Q_2 = -\epsilon$ is of little consequence. Table 3 shows

TABLE 3.—RELATIVE VALUES OF DESIGN CONSTANTS FOR SINGLE-SPAN BRIDGE STRUCTURES

Structure No.	Square span (ft)	Skew		Constants		
		Angle θ	ϵ	Q_2	Q_5	$\frac{L^2}{4}\sum_{o}^{n}\frac{1}{I}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
(a) RIGID-FRAME BRIDGES						
1	52	47° 50′	1.10	−1.11	1/23,740	1/23,685
2	75	41° 00′	0.8693	−0.8735	1/ 1,075	1/ 1,074
3	56	43° 17′ 20″	0.9420	−0.9521	1/ 1,362	1/ 1,329
(b) ELLIPTICAL ARCH BRIDGES						
4	56.1	45°	1.000	−1.0424	....	....
5	57.3	36° 34′ 33″	0.742	−0.771	....	....

the relative values of these and other constants for three rigid-frame bridges and two elliptical arches.

Therefore, without significant error,

$$Q_2 = \frac{\delta_{ab}}{\delta_{bb}} = -\epsilon \quad \text{(15)}$$

Substituting Eqs. 13, 14, and 15 into Eqs. 11 and 12,

$$A = Q_4\,(-\Delta_{a,R} - \Delta_{a,T} - \epsilon\,\Delta_b + Q_3\,\Delta_d) \quad \text{(16a)}$$

$$B = \epsilon\,A - Q_5\,\Delta_b \quad \text{(16b)}$$

$$C = -Q_6\,\Delta_c \quad \text{(16c)}$$

and

$$D = -\epsilon\,Q_3\,A - Q_7\,\Delta_d \quad \text{(16d)}$$

Substituting Eqs. 3 into Eq. 4b the result is the following very useful equation:

$$M_v = M_{ov} - (B - \epsilon A)\, u - C \sin\phi + (D + \epsilon A\, h) \cos\phi - \epsilon A\, y \cos\phi \ldots (17)$$

The structural significance of the foregoing mathematical treatment may be better understood by inspection of Fig. 2(*a*). The redundant A' is here shown as turned through the angle θ, and acts along the longitudinal axis of the structure. The redundant A' is really the resultant of A and ϵA with a slight excess or deficiency. Eq. 16b means that part of force B has been used thus to turn A. The foregoing treatment isolates the effect of each redundant on the torsional moment M_v. This isolation is necessary in order to evaluate the contribution of each redundant to the total stress. The writer believes that the algebraic solution is easier to understand than a presentation involving skewed redundants and skewed residual structures.

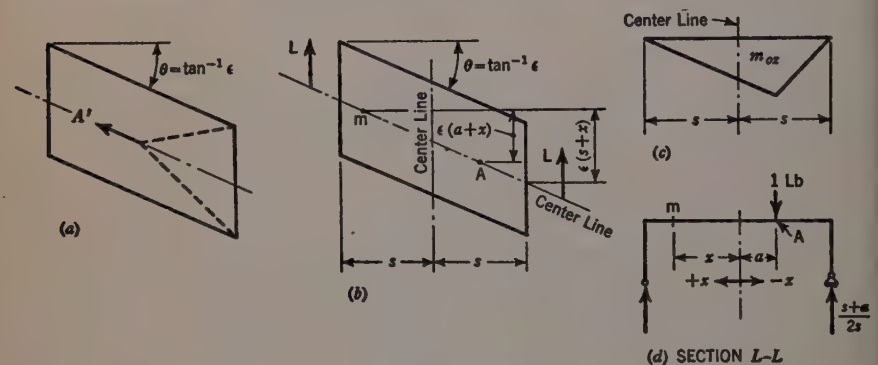

FIG. 2

Vertical Loads.—Fig. 2(*b*) shows a plan of a skewed structure with a unit vertical load applied at point A. The stresses at point m are required. Fig. 2(*d*) shows a square section L-L of the plan in Fig. 2(*b*). The distances from the center line to points A and m are indicated as a and x, respectively. Fig. 2(*c*) indicates the *m* moment diagram of the residual structure due to a unit load at point A, Fig. 2(*d*).

For a unit vertical load at point A,

$$m_{ov} = \frac{s + a}{2\, s} (s + x)\, \epsilon \cos\phi - (1 \cos\phi)\, \epsilon\, (a + x) \ldots\ldots\ldots (18a)$$

or

$$m_{ov} = \epsilon \cos\phi\, m_{oz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18b)$$

and therefore, by superposition,

$$M_{ov} = \epsilon \cos\phi\, M_{oz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

One of the most striking examples illustrating how terms combine or cancel out is shown when Eqs. 3, 16, 18b, and 19 are substituted into Eq. 17, thus: $M_v = M_{ov} + \epsilon A\, x \sin\phi - B\, u - C \sin\phi + D \cos\phi = \epsilon\, M_{oz} \cos\phi + \epsilon A\, x \sin\phi$

$- C \sin \phi + D \cos \phi - (B - \epsilon A) u - \epsilon A (x \sin \phi + y \cos \phi - h \cos \phi)$; or

$$M_v = \epsilon \cos \phi M_z + [- (B - \epsilon A) u + C \sin \phi + (D + \epsilon A h) \cos \phi] \quad (20)$$

For the vault of a rigid frame, in which $\sin \phi = 0$ and $\cos \phi = 1$, for applied loads the three terms within the bracket are of secondary order and their combination is even less significant. Therefore, for the vault—

$$M_v = \epsilon \cos \phi M_z \quad (21)$$

and, for the legs of a rigid frame, in which $\cos \phi = 0$, $\sin \phi = 1$, and $u = \frac{L}{2}$,

$$M_v = - \frac{(B - \epsilon A) L}{2} \quad (22a)$$

Eq. 22*a* may be evaluated from Eq. 16*b* as—

$$M_v = Q_5 \Delta_b \frac{L}{2} \quad (22b)$$

Eq. 21 indicates that influence lines for M_v are not required since they will be the influence lines for M_z to a different scale. For vertical loads, in addition to Eqs. 21 and 22,

$$A = - Q_4 \Delta_{a,R} \quad (23a)$$

$$(B - \epsilon A) = - Q_5 \Delta_b \quad (23b)$$

$$C = - Q_6 \Delta_c \quad (23c)$$

and

$$D + \epsilon Q_3 A = - Q_7 \Delta_d \quad (23d)$$

For symmetrical vertical loads $\Delta_c = 0$; therefore $C = 0$.

For rigid-frame bridges Δ_b is almost equal to zero; therefore $(B - \epsilon A)$ is almost equal to zero (less than 6% of ϵA). Eq. 23*a* is found from Eq. 16*a* in which the combination of terms $(- \Delta_{a,T} - \epsilon \Delta_b + Q_3 \Delta_d)$ is of secondary order

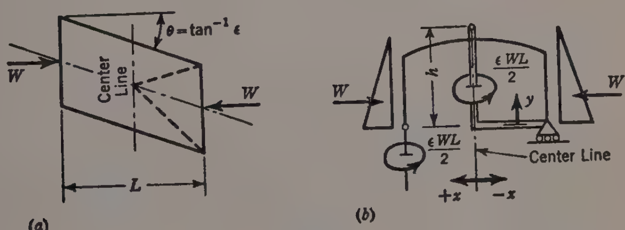

FIG. 3

for vertical loads and is therefore dropped out. The use of elastic weights and torsional elastic weights to determine $\Delta_{a,R}$, $\Delta_{a,T}$, Δ_b, Δ_c, and Δ_d for vertical loads is fully explained in the original paper filed for reference in the Engineering Societies Library[5a] and elsewhere.[6]

[5a] 29 W. 39th St., New York 18, N. Y.

[6] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950.

Earth Pressure; Both Sides.—Fig. 3 shows a residual structure loaded with earth pressure on both sides. The vertical reactions are zero and the structure is in static equilibrium due to the application of the two equilibrating moments $\epsilon\, W\, L/2$, one at the left support and the other at the center line on the rigid bracket. The equilibrating moment is divided into two equal parts, as shown, in order that the loaded residual structure may be symmetrical about the center axis. The moment M_{oz} may be determined by the shear increment method,[6] starting at the bottom of a leg. From the values of M_{oz} the value of $\Delta_{a,R}$ is found. This procedure is part of a standard rectangular analysis which is fully illustrated elsewhere.[6] For the torsional analysis,

$$M_{ov} = M_{oy} \sin \phi \qquad (24a)$$

and

$$M_{oy} = \left(\frac{\epsilon\, W\, L}{2}\right) - \epsilon\, W \left(\frac{L}{2} - x\right) = + \epsilon\, W\, x \qquad (24b)$$

Therefore, for the rectangular elastic system,

$$\Delta_{a,R} = - \Delta_s \Sigma\, M_{oz} \frac{y}{I\, E_c} \qquad (25)$$

and, for the torsional elastic system,

$$\Delta_{a,T} = + \epsilon^2 \Delta_s\, W \sum \frac{x^2 \sin^2 \phi}{F\, E_s} = + W\, \delta_{aa,T} \qquad (26a)$$

$$\Delta_b = - \epsilon\, \Delta_s\, W \sum \frac{u\, x \sin \phi}{F\, E_s} = + W\, \delta_{ab} \qquad (26b)$$

$$\Delta_c = - \epsilon\, \Delta_s\, W \sum \frac{x \sin \phi}{F\, E_s} = + W\, \delta_{ac} = 0 \qquad (26c)$$

and

$$\Delta_d = + \epsilon\, \Delta_s\, W \sum \frac{x \sin \phi \cos \phi}{F\, E_s} = + W\, \delta_{ad} \qquad (26d)$$

In Eqs. 26, F denotes a factor of torsion.

Substituting Eqs. 25 and 26 into Eqs. 16,

$$A = Q_4 \left[- \Delta_{a,R} - W \left(\delta_{aa,T} - \frac{\delta^2_{ab}}{\delta_{bb}} - \frac{\delta^2_{ad}}{\delta_{dd}} \right) \right]$$
$$= Q_4 (- \Delta_{a,R} - W\, Q_1) = - Q_4\, \Delta_{a,R} \qquad (27a)$$

$$B = \epsilon\, A - W \frac{\delta_{ab}}{\delta_{bb}} = \epsilon\, A + \epsilon\, W = \epsilon\, (W + A) \qquad (27b)$$

$$C = 0 \qquad (27c)$$

and

$$D = - \epsilon\, Q_3\, A - W\, Q_7\, \delta_{ad} = - \epsilon\, Q_3\, A - \epsilon\, Q_3\, W = - Q_3\, \epsilon\, (W + A) \qquad (27d)$$

Substituting Eqs. 24 and 27 into Eq. 4b,

$$\epsilon\, W\, x \sin \phi + \epsilon\, A\, x \sin \phi - (W + A)\, u - 0 - Q_3\, (W + A) \cos \phi = M_v \qquad (28)$$

Substituting Eqs. 3 into Eq. 28 and factoring,

$$M_v = \epsilon\,(W + A) \cos \phi \,(h - y - Q_3) \ldots\ldots\ldots\ldots (29a)$$

Applying Eq. 5*a*, and remembering that $V = 0$ for balanced earth pressure, the torsional moment is then

$$M_v = \epsilon\, T_v \,(h - y - Q_3) \ldots\ldots\ldots\ldots (29b)$$

Temperature Change.—Fig. 4 indicates a plan of the residual structure and also the deformed position of the residual structure due to a rise in temperature. Change in width is not important and therefore is not shown. Let c be the coefficient of temperature expansion and t be the temperature change in degrees Fahrenheit. Then, referring to Fig. 4, Δ_c, Δ_d, M_o, and C are each equal to zero, and

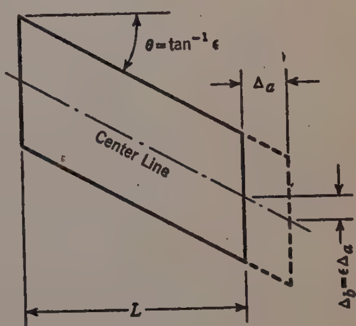

FIG. 4

$$\Delta_a = -\,c\,t\,L \ldots\ldots (30a)$$

and

$$\Delta_b = -\,\epsilon\,c\,t\,L \ldots\ldots (30b)$$

Substituting the foregoing values into Eqs. 16,

$$A = Q_4\,(+\,c\,t\,L + \epsilon^2\,c\,t\,L) = Q_4\,(1 + \epsilon^2)\,c\,t\,L \ldots (31a)$$

$$B = \epsilon\,A + \epsilon\,Q_5\,c\,t\,L \ldots\ldots\ldots\ldots (31b)$$

$$C = 0 \ldots\ldots\ldots\ldots (31c)$$

and

$$D = -\,\epsilon\,Q_3\,A \ldots\ldots\ldots\ldots (31d)$$

Substituting Eqs. 31 into Eq. 18*a*,

$$M_v = -\,\epsilon\,Q_5\,c\,t\,L\,u + \epsilon\,A\,(h - y - Q_3) \cos \phi \ldots\ldots (32a)$$

Applying Eq. 32*a* to arches and rigid frames it has been found that, at points where the torsional moment M_v has any practical significance, the major contribution is from the term $-\,\epsilon\,Q_5\,c\,t\,L\,u$ and that the second term may be neglected. Therefore,

$$M_v = -\,\epsilon\,Q_5\,c\,t\,L\,u \ldots\ldots\ldots\ldots (32b)$$

Summary of Equations; Single Span

Inspection of Eqs. 21, 29*b*, and 32*b* shows that the tangent, ϵ, of the skew angle appears as a parameter (Table 4). Also these equations indicate that no torsional elastic analysis is required because, for any point, a rectangular stress, multiplied by a constant, yields the required torsional stress. For temperature change, a constant, $-\,\epsilon\,(Q_5\,c\,t\,L)$, multiplied by u gives the required torsional moment. Furthermore, Eqs. 27*b* and 31*b* show that the tor-

sional cross shear T_z (= B) is susceptible to the same treatment. The equations necessary for a practical analysis of skewed structures are shown in Table 4.

TABLE 4.—EQUATIONS FOR THE ANALYSIS OF SINGLE-SPAN SKEWED STRUCTURES

Stress	Vertical Loads		Temperature Change		Balanced Earth Pressure	
	Equation	No.	Equation	No.	Equation	No.
M_z	$M_{oz} - A\,y$	4a	$\mp Q_2\,(1 + \epsilon^2)\,c\,t\,L\,y$	31a	$M_{oz} - A\,y$	4a
T_v	$V \sin\phi + A \cos\phi$	5a	$\pm Q_2\,(1 + \epsilon^2)\,c\,t\,L \cos\phi$	5a	$(W + A) \cos\phi$	5a
M_v/ϵ	$M_z \cos\phi$	21	$\mp Q_5\,c\,t\,L\,u$	32b	$T_v\,(h - y - Q_1)$	29b
T_z/ϵ	A	23b	$\pm A \pm Q_5\,c\,t\,L$	31b	$W + A$	27b

Comparisons of exact values and proposed approximations for Q_2 and Q_5 are shown in Table 3. Critical examination of the terms entering the exact evaluation justify the general conclusion that neglected terms are of a minor nature.

ANALYSIS OF FOUNDATION CONDITIONS

Foundation conditions are of particular importance for skewed rigid-frame bridges and arches. In addition to R_x and R_y the foundation design (see Fig. 5) must take into account the effects of R_z, M_x (= $R_y\,e'$), and M_y (= $R_x\,e$), in order to insure the structural action on which the analysis and design is predicated. The foundation restraints that are significant are shown in Table 5.

TABLE 5.—SIGNIFICANT FOUNDATION RESTRAINTS

Foundation restraint[a]	Dead load	Balanced earth pressure	Temperature change[b]
R_x	ΣH	ΣH	$(1 + \epsilon^2)\,\Sigma H$
R_y	Vertical reaction[c]	Vertical reaction[c]	Vertical reaction[c]
R_z	$\epsilon\,R_x$	$\epsilon\,(R_x - \Sigma W)$	$\epsilon\left[R_x + \dfrac{6.7\,c\,t\,E_c}{(1+\mu)\,s\,L\,\Sigma\frac{1}{I}}\right]$
M_x	$-R_z\,y_c$	$-R_z\,y_c$	$-R_z\,y_c$
M_y	Negligible	Negligible	$\epsilon\left[\dfrac{3.35\,c\,t\,E_c}{(1+\mu)\,s\,\Sigma\frac{1}{I}}\right]$
M_z	0	0	0

[a] M_z assumed zero for a hinged structure. [b] See text for limits of $\Sigma 1/I$. [c] Symbol R_y denotes the vertical reaction for the transformed structure plus the vertical effect of the redundants.

The equations are for a unit foot of width which must be multiplied by b in order to obtain the total foundation restraints. Table 5 also shows the foundation restraints that can usually be neglected. (The quantity $\sum\frac{1}{I}$ denotes the

summation for points in both end legs, and excludes values for the vault and for the center leg of the double-span frames.)

Values for live load may be determined by positioning the load for maximum M_z at the crown.

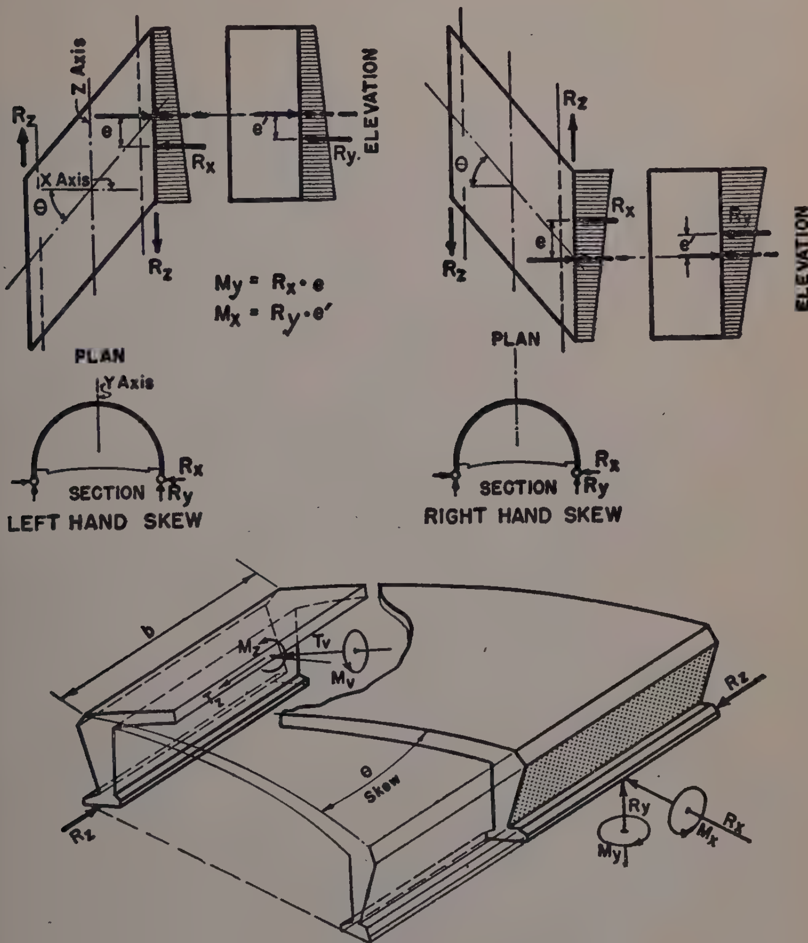

FIG. 5.—ANALYSIS OF FOUNDATION CONDITIONS

(a) Left-Hand Skew
(b) Right-Hand Skew
(c) Isometric Sketch

If the foundation material does not provide full restraint as assumed, it is possible to substitute the partial restraint, as a function of k (the soil modulus), into the equations. This will result in partial fixity about the x-axis and the y-axis but requires the assumption that k is a known and uniform value.

Design of Longitudinal Reinforcement

A simple process for the design of the longitudinal reinforcement is the result of an approach to the problem based on the following propositions:

1. The square-span moments (M_s) and thrusts (T_v) (that is, as computed on the span perpendicular to the abutments) are independent of the skew angle for all applied loads.
2. The square-span stresses for temperature change vary as the square of the secant of the skew angle.
3. Using required square-span areas of steel reinforcement, a simple "turning" process will give the required steel areas for the skew span—that is, for bars running parallel to the skew center line.
4. With a "turned" longitudinal steel pattern, only nominal transverse reinforcement, parallel to the abutments, will be required, except for bridges of very large skew or a very long span. For single-span structures, this limit will be about 50°. This transverse reinforcement will be amply provided for by 1-in.-round rods, 18 in. on centers, as usually shown in details for unskewed structures.

The foregoing propositions for single-span and multiple-span structures have been proved mathematically and verified by comparing the results of the exact and the simplified method of analysis as applied to a series of hypothetical rigid-frame bridges all having the same square span but different skews, to a

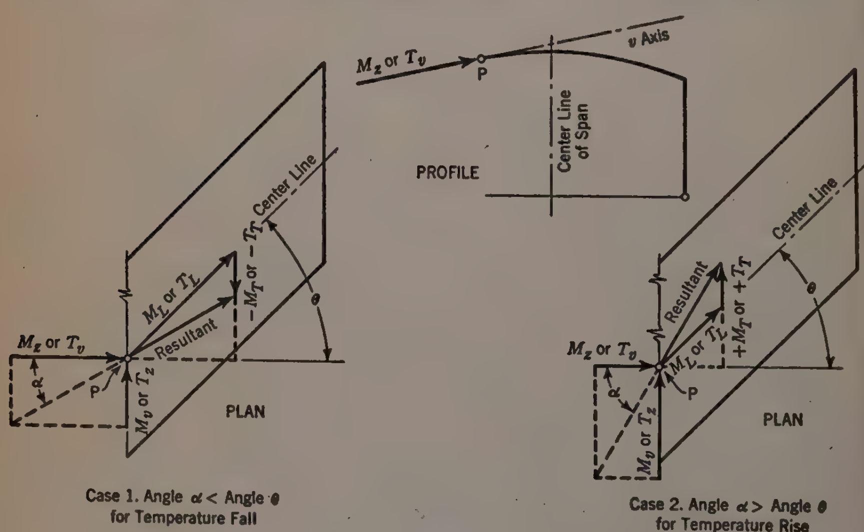

Fig. 6.—Transformations

maximum of 60°. In addition, the propositions have been verified by comparing the results of the simplified method with the results of exact analysis for a number of actual bridges. These proofs are not included in this paper but are filed for reference in the Engineering Societies Library.

Transformations.—Fig. 6 proposes two cases to demonstrate the "turning process" mentioned in proposition 3. In both cases, the following equations apply:

$$M_L \doteq M_z \sec V \qquad (33a)$$

The angle V is the projection of the skew angle θ onto the plane tangent to the neutral axis at the point under consideration:

$$T_L = T_v \sec V \qquad (33b)$$

$$M_T = M_z \tan V - M_v \qquad (34a)$$

and

$$T_T = T_v \tan V - T_z \qquad (34b)$$

In general—

$$\tan V = \epsilon \cos \theta \qquad (35)$$

The reinforcement is found for bending and direct stress by using the equations instead of numerical values. Standard nomenclature used in concrete design

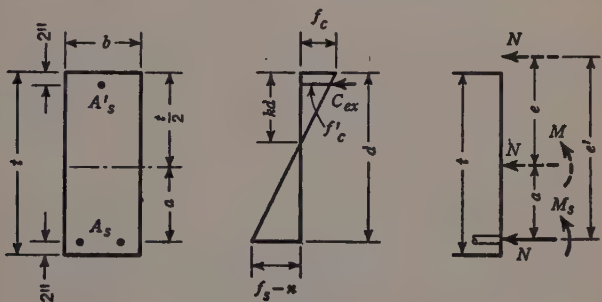

FIG. 7

is employed in the following demonstration: Referring to Fig. 7,

$$e_L = \frac{M_L}{T_L} = \frac{M_z \sec V}{T_v \sec V} = \frac{M_z}{T_v} = e \qquad (36a)$$

and

$$e' = e + a \qquad (36b)$$

Then

$$M_{sL} = e' T_L = e' T_v \sec V = M_{sR} \sec V \qquad (37a)$$

Also

$$A_{sL} = \frac{M_{sL}}{f_s j d} - \frac{T_L}{f_s} = \left(\frac{e' T_v}{f_s j d} - \frac{T_v}{f_s}\right) \sec V = A_{sR} \sec V \qquad (37b)$$

In Eq. 37b, A_{sR} is the steel area (Fig. 8) before turning and the spacing of the bars must be measured parallel to the abutment. This is logical because A_{sR} must be increased to be equally effective in resisting M_{sR} when the bar is turned from its most effective direction. If the spacing perpendicular to the bars (that is, perpendicular to the face) is more desirable, either this spacing must be multiplied by $\cos \theta$, or the steel area must be multiplied by $\sec \theta$.

Therefore, for a pattern in which the main steel is parallel to the face of the bridge and in which the spacing is measured perpendicular to the bars,

$$A_{sL} = A_{sR} \sec V \sec \theta \qquad (38)$$

In the usual rigid frames sec V = sec θ for vault sections and sec $V = 1$ for legs. Therefore, for vault sections—

$$A_{sL} = A_{sR} \sec^2 \theta = A_{sR} (1 + \epsilon^2) \qquad (39a)$$

for legs with spacing still measured perpendicular to the face—

$$A_{sL} = A_{sR} \sec \theta \qquad (39b)$$

and, for legs with spacing measured along the abutment—

$$A_{sL} = A_{sR} \qquad (39c)$$

Explanation of the "Turning Process" for Reinforcement.—The area of reinforcing bars in the vault of the frame, as calculated for the square span, is A_{sR} in Fig. 8. The area of the "turned" steel bars A_{sT} must be A_{sR} sec V in order to be equally effective in the square-span direction after turning. If the arching of the vault (top of the frame) is small, V may be assumed, for all practical purposes, equal to θ for points in the vault.

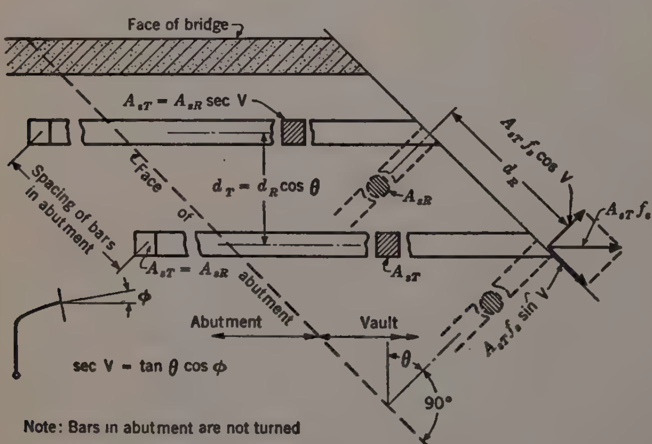

FIG. 8

As shown in Fig. 8, the plan spacing of the "turned" bars d_T, measured perpendicularly to the bars themselves, will be the spacing of the unturned bars multiplied by cos θ. The net result of the two multiplications is that the required steel area in the top of the frame of the skewed bridge is the required area calculated for the square span multiplied by sec V sec θ. At the crown, sec V sec θ will be equal to sec² θ, which for all practical purposes may also be applied to the vault of rigid-frame bridges.

Turning the steel pattern parallel to the skew center line reduces its effectiveness in the square-span direction but increases its effectiveness in the transverse direction (parallel to the abutments), as seen from Fig. 8. The resolution of the stress in any bar into two components is indicated, one component being effective in resisting bending moment about the z-axis and the other being effective in resisting the cross shear T_z and the torsional moment M_v. This is analogous to the wedge-shaped beam in which one component of the stress in the bars resists bending and the other component resists shear.[7]

The effectiveness of the "turned" reinforcement in the transverse direction, combined with the effectiveness of the concrete in shear and torsion, and the effectiveness of the nominal transverse reinforcement, are sufficient to take care of all torsional effects resulting from skew, to a maximum skew angle of about 50° for the single-span structure.

The increase in steel areas contingent on the "turning" process does not apply to points in the vertical legs. For these points, the steel areas as calculated for the square span (Eq. 39c) are correct for the skew structure, with the spacing of bars as discussed.

Extension to Double-Span Bridges

A practical analysis results if propositions 1, 2, 3, and 4, which were developed for single-span structures, are applied to double-span skewed structures. To justify this application consider Fig. 9 which shows a double-span

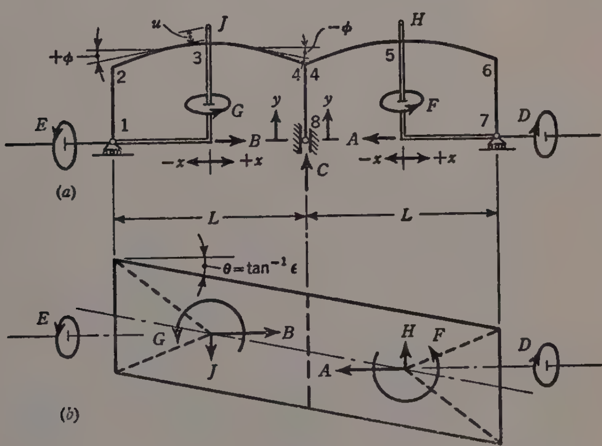

Fig. 9

skewed rigid frame. This structure may be replaced by a single-span structure with a center post considered as a flexible bracket. A system of loads is applied to the end of this bracket and these loads are of such value that they exactly replace the reactions of the original structure. This transformation makes logical the application of the principles of single-span analysis to double-

[7] "Concrete Engineers' Handbook," by George A. Hool and Nathan C. Johnson, McGraw-Hill Book Co., Inc., New York, N. Y., 1919, p. 314.

span skewed structures. The validity of this logical application has been checked by analyzing, independently, the rectangular elastic system for the rectangular redundants for a unit vertical influence load and by comparing the answers with those of the precise analysis. The precise analysis consisted of the solution of two sets of simultaneous equations. Each set contained six elastic equations, and the analysis is similar to the method proposed by Mr. Rathbun[8] in 1931, modified to apply to hinged foundations.

A simpler analysis of the double-span structure is indicated in the following demonstration. Fig. 9 shows the most advantageous residual structure because it is symmetrical.

TABLE 6.—MOMENT AT A GIVEN SECTION DUE TO EACH UNIT REDUNDANT

Section	$A = +1$		$B = +1$		$C = +1$		$D = +1$	$E = +1$	$F = +1$	$G = +1$	$H = +1$	$J = +1$
	m_x	m_y	m_x	m_y	m_x	m_y	m_y	m_y	m_y	m_y	m_y	m_y
1–2	0	0	$-y$	$-\epsilon x \sin\phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	0	0	$+\cos\phi$	0	$+\sin\phi$	0	$-u$
2–3	0	0	$-y$	$-\epsilon x \sin\phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$-\frac{\epsilon}{2} x \cos\phi$	0	$+\cos\phi$	0	$+\sin\phi$	0	$-u$
3–4	0	0	$-y$	$-\epsilon x \sin\phi$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$-\frac{\epsilon}{2} x \cos\phi$	0	$+\cos\phi$	0	$+\sin\phi$	0	$-u$
4–8	$+y$	$-\epsilon x \sin\phi$	$-y$	$-\epsilon x \sin\phi$	0	0	$+\cos\phi$	$+\cos\phi$	$-\sin\phi$	$+\sin\phi$	$-u$	$-u$
4–5	$-y$	$-\epsilon x \sin\phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$-\frac{\epsilon}{2} x \cos\phi$	$+\cos\phi$	0	$-\sin\phi$	0	$-u$	0
5–6	$-y$	$-\epsilon x \sin\phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$-\frac{\epsilon}{2} x \cos\phi$	$+\cos\phi$	0	$-\sin\phi$	0	$-u$	0
6–7	$-y$	$-\epsilon x \sin\phi$	0	0	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	0	$+\cos\phi$	0	$-\sin\phi$	0	$-u$	0

Table 6, similar to Table 1, shows the moments at each section of the structure due to each unit redundant; and a tabulation similar to Table 2 may be easily developed from it, showing the unit deformations for each unit redundant. The location of the sections is shown in Fig. 9. For the general case, there are nine redundants and nine simultaneous equations, which are as follows:

$$\delta_a = 0 = A\,\delta_{aa} + B\,\delta_{ab} + C\,\delta_{ac} + D\,\delta_{ad} + E\,\delta_{ae} + F\,\delta_{af} + G\,\delta_{ag} + H\,\delta_{ah} + J\,\delta_{aj} + \Delta_a \quad \text{(40a)}$$

$$\delta_b = 0 = A\,\delta_{ba} + B\,\delta_{bb} + C\,\delta_{bc} + D\,\delta_{bd} + E\,\delta_{be} + F\,\delta_{bf} + G\,\delta_{bg} + H\,\delta_{bh} + J\,\delta_{bj} + \Delta_b \quad \text{(40b)}$$

$$\delta_c = 0 = A\,\delta_{ca} + B\,\delta_{cb} + C\,\delta_{cc} + D\,\delta_{cd} + E\,\delta_{ce} + F\,\delta_{cf} + G\,\delta_{cg} + H\,\delta_{ch} + J\,\delta_{cj} + \Delta_c \quad \text{(40c)}$$

$$\delta_d = 0 = A\,\delta_{da} + B\,\delta_{db} + C\,\delta_{dc} + D\,\delta_{dd} + E\,\delta_{de} + F\,\delta_{df} + G\,\delta_{dg} + H\,\delta_{dh} + J\,\delta_{dj} + \Delta_d \quad \text{(40d)}$$

[8] "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 1.

$$\delta_e = 0 = A\,\delta_{ea} + B\,\delta_{eb} + C\,\delta_{ec} + D\,\delta_{ed} + E\,\delta_{ee} + F\,\delta_{ef} + G\,\delta_{eg} + H\,\delta_{eh} + J\,\delta_{ej} + \Delta_e \quad (40e)$$

$$\delta_f = 0 = A\,\delta_{fa} + B\,\delta_{fb} + C\,\delta_{fc} + D\,\delta_{fd} + E\,\delta_{fe} + F\,\delta_{ff} + G\,\delta_{fg} + H\,\delta_{fh} + J\,\delta_{fj} + \Delta_f \quad (40f)$$

$$\delta_g = 0 = A\,\delta_{ga} + B\,\delta_{gb} + C\,\delta_{gc} + D\,\delta_{gd} + E\,\delta_{ge} + F\,\delta_{gf} + G\,\delta_{gg} + H\,\delta_{gh} + J\,\delta_{gj} + \Delta_g \quad (40g)$$

$$\delta_h = 0 = A\,\delta_{ha} + B\,\delta_{hb} + C\,\delta_{hc} + D\,\delta_{hd} + E\,\delta_{he} + F\,\delta_{hf} + G\,\delta_{hg} + H\,\delta_{hh} + J\,\delta_{hj} + \Delta_h \quad (40h)$$

$$\delta_j = 0 = A\,\delta_{ja} + B\,\delta_{jb} + C\,\delta_{jc} + D\,\delta_{jd} + E\,\delta_{je} + F\,\delta_{jf} + G\,\delta_{jg} + H\,\delta_{jh} + J\,\delta_{jj} + \Delta_j \quad (40j)$$

For symmetrical structures, the dead load, the balanced earth pressure load, and the temperature change effects are symmetrical. Only the applied live load can be unsymmetrical. It is obvious from Fig. 9 that for a symmetrical, double-span structure subjected to symmetrical loads the redundants are equal in pairs, as follows:

$$A = B \quad (41a)$$

$$D = E \quad (41b)$$

$$F = G \quad (41c)$$

and

$$H = J \quad (41d)$$

Fig. 10 shows the residual structure with the applied symmetrical redundants.

Eqs. 41 indicate, from symmetry, that the center post is subjected only to the vertical rectangular reaction C and that no torsional stresses exist for the center post sections. Thus, for symmetry, the nine elastic equations (Eqs. 40) are reduced to five elastic equations (Eqs. 43) written for five redundants, two being rectangular and three torsional.

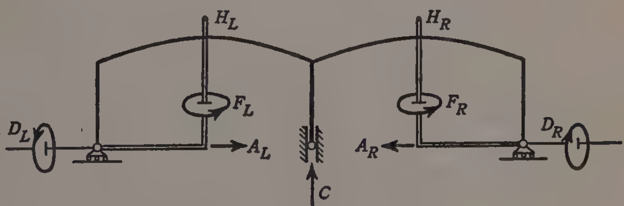

FIG. 10

If Eq. 40a is added to Eq. 40b, Eq. 40d to Eq. 40e, Eq. 40f to Eq. 40g, and Eq. 40h to Eq. 40j, five elastic equations may be written for symmetrical structures symmetrically loaded. If the conjugate redundants from Eqs. 41 are substituted into the five equations, and the conjugate deformations of

Table 7 are also substituted,

$$Q_{12}\,A + Q_{13}\,C + Q_{14}\,D + Q_{15}\,F + Q_{16}\,H = -\Delta_a \quad \text{(42a)}$$

$$Q_{13}\,A + Q_{17}\,C + Q_{18}\,D + Q_{19}\,F + Q_{20}\,H = -\Delta_c \quad \text{(42b)}$$

$$Q_{14}\,A + Q_{18}\,C + Q_{21}\,D + Q_{22}\,F + Q_{23}\,H = -\Delta_d \quad \text{(42c)}$$

$$Q_{15}\,A + Q_{19}\,C + Q_{22}\,D + Q_{24}\,F + Q_{25}\,H = -\Delta_f \quad \text{(42d)}$$

and

$$Q_{16}\,A + Q_{20}\,C + Q_{23}\,D + Q_{25}\,F + Q_{26}\,H = -\Delta_h \quad \text{(42e)}$$

In Eqs. 42 the conjugate deformations are written from the same formula and the sum of any two conjugate deformations may be found by including the new sections—that is, summing up within the new limits. Application of the logical propositions (substantiated by comparisons and tests) permits the separate determination of the rectangular and torsional redundants. This separation results in a practical solution whereas the simultaneous solution of the five equations results in a precise solution. The logical conclusion is that the same precision exists for the double span as exists for the single span.

If the redundant A and the redundant C are found independently from the rectangular elastic analysis, Eqs. 42 may be written as follows:

$$Q_{21}\,D + Q_{22}\,F + Q_{23}\,H = -\Delta_d - Q_{14}\,A - Q_{18}\,C \quad \text{(43a)}$$

$$Q_{22}\,D + Q_{24}\,F + Q_{25}\,H = -\Delta_f - Q_{15}\,A - Q_{19}\,C \quad \text{(43b)}$$

and

$$Q_{23}\,D + Q_{25}\,F + Q_{26}\,H = -\Delta_h - Q_{16}\,A - Q_{20}\,C \quad \text{(43c)}$$

Thus only three simultaneous equations are required to be solved for an exact analysis of the double-span symmetrical skewed structure with symmetrical loading. Of course, this analysis is not required unless the critical skew has been exceeded. It is estimated that more than 60% of all double-span, skewed frames and arches have less than critical skew.

Live load may be applied as an unsymmetrical load, but by a transformation it is possible to convert the unsymmetrical applied load into two elements which are symmetrical. Consider Fig. 11 which shows an unsymmetrical load P applied at point m. Fig. 11(a) shows a pair of loads $P/2$, applied symmetrically. Fig. 11(b) shows another pair of loads, $P/2$ and $-P/2$ applied as shown. Combining the loadings in Figs. 11(a) and 11(b), the result is exactly the unsymmetrically applied load P of Fig. 11(c).

This transformation is particularly useful when the analysis is one of the moment distribution methods. Such methods usually distribute 100 units or 1,000 units of moment applied at the center joint, and this distribution will quickly yield the required values for the moment, $P\,x$, applied to the center joint.

Numerous approximations have been uncovered by isolating and analyzing individual elements. Among the more useful approximations are the following, which eliminate considerable computations.

It has been observed that the element $(h - y - Q_3)$ in Eq. 29b may be approximated as $(y_c - y)$, in which y_c is the y-value for the crown. This

TABLE 7.—SCHEDULE OF CONJUGATE DEFORMATIONS

Eq. 40	Coefficient For:	Coefficient No.	Formula Element Nos.: 1	2	3	4	Formula	Center Leg Values for Elements: Formula	1	2	3	4	Σ
(a) SECTIONS 1-2-3-4-5-6-7, FIG. 9(a), CONTRIBUTING TO TOTAL VALUE													
a, b	A	$Q_{12} =$	$\delta_{aa} +$	$\delta_{bb} +$	$\delta_{ab} +$	δ_{ba}	$\pm \Sigma (G_1 y^2)$	$\pm \Sigma (G_1 y^2)$	+	+	−	−	0
							$+ \epsilon^2 \Sigma (G x^2) \sin^2 \phi$	$\epsilon^2 \Sigma (G x^2)$	+	+	−	−	0
a, b c, c	$\left.\begin{matrix}C\\A\end{matrix}\right\}$	Q_{13}	δ_{ac}	δ_{bc}			$\pm \frac{1}{2} \Sigma \left[G_1 y \left(\frac{L}{2} + x \right) \right]$	$\mp \frac{1}{2} \Sigma (G_1 y)$	+	−			0
							$+ \frac{\epsilon^2}{2} \Sigma \left[G x \left(\frac{L}{2} + x \right) \sin \phi \cos \phi \right]$	0	0	0			0
a, b d, e	$\left.\begin{matrix}D\\A\end{matrix}\right\}$	Q_{14}	δ_{ad}	δ_{bd}	δ_{ae}	δ_{be}	$- \epsilon \Sigma (G x \sin \phi \cos \phi)$	0	0	0	0	0	0
a, b f, g	$\left.\begin{matrix}F\\A\end{matrix}\right\}$	Q_{15}	δ_{af}	δ_{bf}	δ_{ag}	δ_{bg}	$\pm \epsilon \Sigma (G x \sin^2 \phi)$	$\pm \epsilon \Sigma (G x)$	+	−	+	−	0
a, b h, j	$\left.\begin{matrix}H\\A\end{matrix}\right\}$	Q_{16}	δ_{ah}	δ_{bh}	δ_{aj}	δ_{bj}	$\epsilon \Sigma (G u x \sin \phi)$	$\pm \epsilon \frac{L^2}{4} \Sigma G$	+	−	+	−	0
c, e	C	Q_{17}	δ_{ce}				$\frac{1}{4} \Sigma \left[G_1 \left(\frac{L}{2} + x \right)^2 \right]$	0	0				0
							$+ \frac{\epsilon^2}{4} \Sigma \left[G \left(\frac{L}{2} + x \right)^2 \cos^2 \phi \right]$	0	0				0
c, c d, e	$\left.\begin{matrix}D\\C\end{matrix}\right\}$	Q_{18}	δ_{cd}	δ_{ce}			$- \frac{\epsilon}{2} \Sigma \left[G \left(\frac{L}{2} + x \right) \cos^2 \phi \right]$	0	0	0			0
c, c f, g	$\left.\begin{matrix}F\\C\end{matrix}\right\}$	Q_{19}	δ_{cf}	δ_{cg}			$\pm \frac{\epsilon}{2} \Sigma \left[G \left(\frac{L}{2} + x \right) \sin \phi \cos \phi \right]$	0	0	0			0
c, c h, j	$\left.\begin{matrix}H\\C\end{matrix}\right\}$	Q_{20}	δ_{ch}	δ_{cj}			$+ \frac{\epsilon}{2} \Sigma \left[G u \left(\frac{L}{2} + x \right) \cos \phi \right]$	0	0	0			0
d, e	D	Q_{21}	δ_{dd}	δ_{ee}	δ_{ed}	δ_{de}	$\Sigma (G \cos^2 \phi)$	0	0	0	0	0	0
d, e f, g	$\left.\begin{matrix}F\\D\end{matrix}\right\}$	Q_{22}	δ_{df}	δ_{ef}	δ_{dg}	δ_{eg}	$\pm \Sigma (G \sin \phi \cos \phi)$	0	0	0	0	0	0
d, e h, j	$\left.\begin{matrix}H\\D\end{matrix}\right\}$	Q_{23}	δ_{dh}	δ_{eh}	δ_{dj}	δ_{ej}	$- \Sigma (G u \cos \phi)$	0	0	0	0	0	0
(b) SECTIONS 1-2-3-4-8 AND 8-4-5-6-7, FIG. 9(a), CONTRIBUTING TO TOTAL VALUE													
f, g	F	Q_{24}	δ_{ff}	δ_{gg}	δ_{gf}	δ_{fg}	$\Sigma (G \sin^2 \phi)$	ΣG	+	+	+	+	+
f, g h, j	$\left.\begin{matrix}H\\F\end{matrix}\right\}$	Q_{25}	δ_{fh}	δ_{gh}	δ_{fj}	δ_{gj}	$\Sigma (G u \sin \phi)$	$\Sigma (G u)$	+	+	+	+	+
h, j	H	Q_{26}	δ_{hh}	δ_{jj}	δ_{jh}	δ_{hj}	$\Sigma (G u^2)$	$\Sigma (G u^2)$	+	+	+	+	+

eliminates the computation for h and Q_3. By inspection of Eq. 14e and Table 2, it is observed that the legs of a rigid frame contribute almost the entire value of Q_5 and therefore, with little error,

$$Q_5 = \frac{L^2}{4} \sum \frac{1}{F} = \frac{3.58\, L^2}{48} \sum \frac{1}{I} \qquad (44)$$

(The constant multiplier is not shown in Eq. 44. The summation is for the legs only.)

This method of analysis may be extended to skewed structures of more than two spans. The method has given satisfactory results for a triple-span, skewed, rigid-frame parkway bridge.

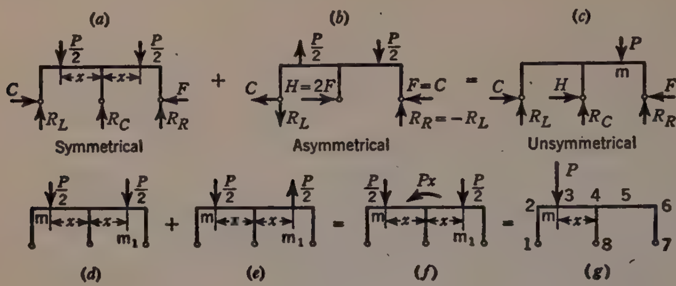

Fig. 11

The free body diagram shown in Fig. 12 is the same for either single-span or multiple-span structures, and therefore any equation written for this free body is valid for all structures. As a result the stress equations and the design

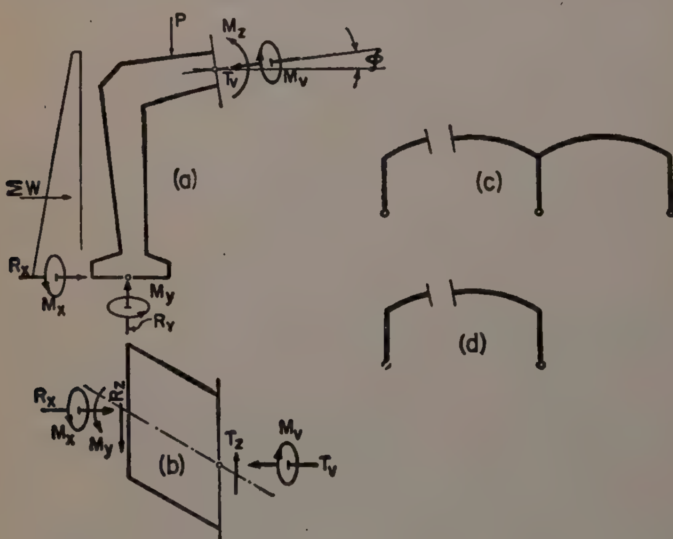

Fig. 12

procedure for the single-span skewed structure may be applied to multiple-span skewed structures.

Test for Limit of Skew Angle with Nominal Transverse Reinforcement

It has been stated that, if the longitudinal reinforcement is designed parallel to the face of the bridge, only nominal reinforcement in the transverse direction

(parallel to the abutments) will be required. This statement will now be examined, and its limits will be determined by analyzing each term that enters into the design of the transverse reinforcement. The critical section is at the crown.

The design moments, M_T, and cross shears, T_T, are found from the transformations. For all the applied loads, M_T and T_T, for the crown section, are negligible. The stresses due to temperature change (see Table 8) are the only

TABLE 8.—STRESSES FOR DESIGN OF TRANSVERSE REINFORCEMENT AT THE CROWN SECTION

Loading	Design moments, M_T	Cross shears,[a] T_T
Dead load	$\epsilon M_x - M_y$	$\epsilon T_y - T_x$
Live load plus impact	$\epsilon M_x - M_y$	$\epsilon T_y - T_x$
Temperature change	$\epsilon R_x y_c$	$\epsilon \dfrac{6.7\, c\, t\, E_c}{(1+\mu)\, s\, L\, \Sigma 1/I}$

[a] See text for limits of $\Sigma 1/I$.

significant stresses. The values from Table 5 are used to determine the crown stresses M_T and T_T.

The next step is to find the unit stresses for the crown section due to M_T and T_T. The unit shearing stress v_t due to the torsion moment M_T may be found from the relation (see Fig. 13):

$$v_t = k \frac{M_T}{b\, t^2} \qquad (45)$$

in which the constant k has been determined variously as shown in Fig. 13. For values of b/t greater than 6, the Saint Venant expression for k is more accurate

TABLE 9.—DESIGN OF TRANSVERSE REINFORCEMENT (SEE FIG. 14)

Point	t (in.)	Torsion		Direct Shear		Geometric Data							Steel (Sq In.)	
		M_T (kip-ft)	v_t	T_T (kips)	v_s	$v_t + v_s$	v	a_1	a_2	a_3	$a_2 + 2a_1$	$a_1 - 3$ (in.)	A_{s1}	A_{s2}
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	27	11.0	45	11.9	55	100	....	....	....	....	....	....	....	....
2	32	11.0	32	11.9	47	79	....	....	....	....	....	....	....	....
3	37	11.0	24	11.9	40	64	....	....	....	....	....	....	....	....
4	36.5	59.2	134	0.54	2	136	24	18.5	3.3	15.2	52.2	15.5	0.02	0.02
5	27	62.0	255	0.54	2	257	145	13.6	7.7	5.9	33.1	10.6	0.31	0.30
6	20	64.0	480	0.54	3	483	371	10.0	7.7	2.3	22.3	7.0	0.81	0.76
....[a]	15	9.0	120	1.63	14	134	22	8.4	1.4	7.0	23.8	4.0	0.016	0.008
11	24	64.2	338	0.54	3	341	229	12.1	8.1	4.0	28.2	9.1	0.51	0.50
12	34	63.0	163	0.54	2	165	53	17.2	5.5	11.7	46.1	14.2	0.08	0.08
13	48	60.8	79	0.54	1	80	....	....	....	....	....	....	....	....

[a] Center-line point for single span. Other points for double-span bridge.

than the Merriman value of 4.5. The latter is applicable only for values of b/t less than 2. In general, for barrel arches and rigid-frame bridges, the ratio of b/t is always greater than 16, and $k = 3$, in Eq. 45, is sufficiently accurate. The cross

shear T_T produces unit shearing stresses (v_s) which are distributed parabolically with a maximum of 1.5 times the average stress. The formulas and numerical values for several typical design sections are shown in Table 9. The allowable unit stress, v_c, in the concrete (see Fig. 14) is taken as 112 lb per sq in. which is

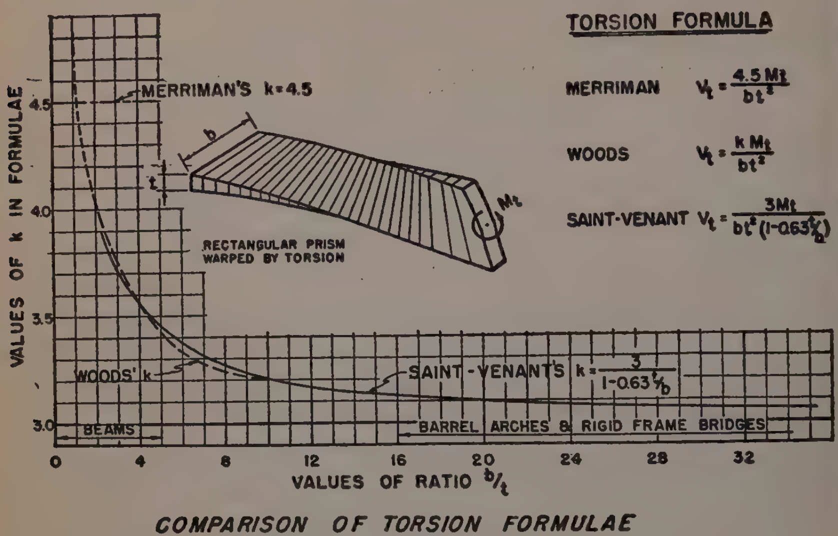

Fig. 13.—Comparison of k-Values in Torsion Formulas

the normal unit stress of 90 lb per sq in. increased by 25% (90 × 1.25) because temperature stress is included. The area A_{s1} is found by equating the moment of the shear wedge $a\,b\,c$, Fig. 14, about the neutral axis, to the moment of $A_{s1} f_s$. The area A_{s2} is found by equating the shear wedge to $A_{s2} f_s$. Referring to Fig. 14, the respective columns in Table 9 are computed by the following formulas: In Col. 3—

$$v_t = \frac{3\,M_T}{12\,t^2} \qquad (46a)$$

In Col. 5—

$$v_s = \frac{1.5\,T_T}{12\,t} \qquad (46b)$$

In Col. 7—

$$v = v_t + v_s - v_c \qquad (46c)$$

In Col. 8—

$$a_1 = \frac{v_t + v_s}{2\,v_t}\,t \qquad (47a)$$

In Col. 9—

$$a_2 = \frac{v_t + v_s - v_c}{2\,v_t}\,t \qquad (47b)$$

In Col. 10—

$$a_3 = a_1 - a_2 \qquad (47c)$$

In Col. 13—

$$A_{s1} = \frac{2\,v\,a_2\,(a_3 + 2\,a_1)}{f_s\,(a_1 - 3\text{ in.})} \qquad (48a)$$

and, in Col. 14—

$$A_{s2} = \frac{6\,v\,a_2}{f_s} \qquad (48b)$$

In Eqs. 48, $f_s = 18{,}000 \times 1.25 = 22{,}500$ lb per sq in. The transverse steel area used is the larger of the two areas, and it is used in both faces.

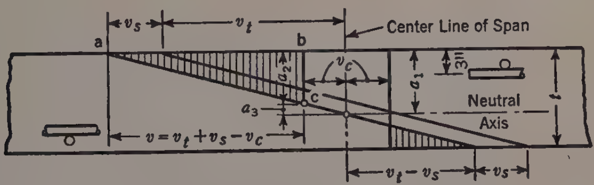

FIG. 14.—TRANSVERSE REINFORCEMENT IN A DOUBLE-SPAN STRUCTURE

Codes and specifications for bridges do not impose a limit for the maximum unit shear due to torsion, but the Bureau of Yards and Docks, United States Department of the Navy, has a commendable standard[9] which treats the subject of combined shears due to torsion and direct shear for beams in flat slab design. For flat slabs, beams at walls and openings, torsion shear plus direct shear, web reinforcement, and longitudinal bars specially anchored, the maximum allowable unit shearing stress for 3,000-lb concrete is 450 lb per sq in. With a 25% increase for the inclusion of temperature stresses, the maximum allowable unit shearing stress would be 450 × 1.25 = 562 lb per sq in. This maximum unit shearing stress is recommended for rectangular sections in the range of these bridges.

By substituting the maximum allowable shearing stress into the equations, the maximum skew for the bridge may be found. Other considerations lead to the tentative conclusion that the limit of skew as determined by the maximum unit shearing stress is close to the limit determined by nominal transverse reinforcement (0.50 sq in. per ft). A more complete demonstration and several numerical examples[6] have been filed for reference in the Engineering Societies Library.[5a]

CONCLUSIONS

It has been demonstrated that for all practical design purposes:

1. With few exceptions, the rectangular elastic system is independent of the torsional elastic system;
2. The rectangular redundants and the rectangular stresses for all applied loads are independent of the torsional elastic system, and independent of the skew angle;
3. Torsional redundants and torsional stresses for applied loads are a linear function of the tangent of the skew angle;

[9] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., Washington, D. C., 1929, Section 8–13d.

4. Conclusions 1, 2, and 3 are true for all the usual shapes of skewed, single-span arches or double-span arches and rigid-frame bridges, and also true for all applied loads regardless of the direction or the point of application of the load; and

5. The rectangular redundants due to temperature change vary as the $\sec^2 \theta$—that is, as $(1 + \epsilon^2)$;

In addition to conclusions 1 to 5 the following important conclusions may be written for the practical application of the method:

6. The torsional stresses may be written directly from the rectangular stresses by multiplying an appropriate rectangular stress by an appropriate geometric or elastic constant, thereby completely eliminating the torsional elastic analysis;

7. The torsional stresses may be written with ϵ as a parameter and thereby represent the torsional stresses for a family of skewed structures;

8. All rectangular and torsional stresses, other than those due to volume change or foundation yield, are independent of Poisson's ratio (that is, $E_c/E_s = \lambda$, since $\lambda = 2(1 + \mu)$);

9. All torsional stresses, other than those due to volume change or foundation settlement, are independent of the formula used to compute the value of F (or r), providing that F is a linear function of b, the width of the structure, and further providing that F is independent of the ratio of b/t within the applied range;

10. Any method that will determine the rectangular stresses satisfactorily may be used as the basis for determining the torsional stresses (such other methods include moment distribution and slope deflection);

11. Inasmuch as Δ_s is a parameter in the demonstration, the analysis is valid for all span lengths;

12. For practical office procedure no torsional analysis is required, and the longitudinal steel may be determined from the rectangular design using the "turning process"; and

13. For extreme skews, long spans, or multiple spans the shearing stresses due to torsion become important and the test for limitations should be applied.

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APPENDIX. NOTATION

The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Structural Analysis (ASA—Z10.8—1949) prepared by a Sectional

Committee of the American Standards Association, with ASCE participation, and approved by the Association in 1949. In general, point P designates any spot on the neutral surface of an arch ring which is halfway between the spandrel walls. The neutral surface is assumed to be midway between the extrados and the intrados.

A subscript R identifies the main symbol with the rectangular elastic system and T, similarly, with the torsional elastic system or in transverse direction, as demonstrated in the text.

A = area; cross-sectional area as further defined by subscripts, s, denoting "steel reinforcement" and, c, denoting "concrete";

A = one of four redundants (A, B, C, and D) introduced in the design of a single-span structure as shown in Fig. 1 and one of nine redundants (A, B, C, D, E, F, G, H, and J) introduced in the design of a double-span skewed structure as shown in Fig. 8; A' = a resultant of redundant A and ϵA in Fig. 2(a); and subscripts L and R denote "left" and "right," respectively;

a = an x-distance measured from the center line of a beam to the point of application of a unit load (see Fig. 2(d));

a = design constants in Table 9, distinguished by numerical subscripts and defined by Eqs. 47 (see also Figs. 7 and 14);

B = a redundant design factor (see under A);

b = width of an arch ring measured parallel to the z-axis (on the skew) (Figs. 1, 5, and 13);

C = a redundant design factor (see under A);

c = coefficient of thermal expansion (Eqs. 30) = 0.0000065;

D = a redundant design factor (see under A);

d = effective depth of a concrete section (see Fig. 7);

E = modulus of elasticity, with subscripts c and s to identify "tension and compression" and "shear," respectively;

E = a redundant design factor (see under A);

e = eccentricity of reaction R_x; and e' = eccentricity of reaction R_y (Fig. 5 and Eqs. 36);

e = eccentricity of thrust, T_v, in Eq. 36a;

F = the factor of torsion (Eqs. 7);

F = a redundant design factor (see under A);

f = fiber stress in reinforced concrete design, the subscripts s and c identifying "steel" and "concrete," respectively;

G = a redundant design factor (see under A);

G = an elastic element defined as $1/F$; G_1 = an elastic element defined as $1/I$;

H = a redundant design factor (see under A);

h = the vertical distance from the x-axis to the elastic center at which the redundant B is applied;

I = moment of inertia of the area of the section cut by a radial plane through point P and parallel to the axis of the barrel (z-axis), about a horizontal line through point P and lying in the radial plane;

J = a redundant design factor (see under A);
j = proportion of effective depth;
k = soil modulus; soil compression constant—that is, the resisting force, in kips, exerted by the soil on a surface 1 ft square which is displaced normally a distance of 1 in.;
k = constant in torsion formula (Eq. 45) (see also Fig. 13);
L = span of the neutral surface measured on the square;
M = bending moment; M_v and M_z are the components of the moment acting on the surface cut by a radial plane through point P about the v-axis and the z-axis, respectively (M_v being positive when it causes the obtuse angle of the section on which it is acting to deflect toward the center of curvature, and M_z being positive when it produces compression in the extrados fibers); M_v is the torsional moment (M_u in Fig. 1 and M_x and M_y in Fig. 5);
m = moment resulting from a unit load applied at a point m; m_v, m_z, m_{ov}, and m_{oz} are the components of the moments corresponding to M_v, M_z, M_{ov}, and M_{oz} when a unit load is applied at point m;
n = a number, such as "indeterminate in the nth degree," or "n equations";
P = applied point load;
Q = substitution factor, with appropriate numerical subscript, function of δ (Eqs. 14);
R = resultant forces or reaction identified by appropriate subscripts x, y, z, etc., to a given axis;
r = an elastic element in Eq. 8;
Δs = the length of an elementary parallelopiped measured, on the square, along the neutral surface (see Fig. 1); and ds is the differential length of an elementary parallelopiped measured, on the square, along the neutral surface;
s = one half the span length, on the square, of an arch (Fig. 2);
T = resultant thrust on a section along an axis identified by appropriate subscript;
t = thickness (Figs. 1 and 13 and Eqs. 45 and 46);
t = temperature rise (+) or temperature fall (−), in degrees Fahrenheit (Eqs. 30);
u = the coordinate of the elastic center referred to the tangent and radial planes through point P and parallel to the plane $z = 0$, u being parallel to the radial plane (the sign is fixed by Eq. 3b);
u_o = a dimension defined as $x \sin \phi + y \cos \phi$ (Fig. 1);
V = projection of the skew angle θ onto the plane tangent to the neutral axis at the point under consideration;
v = unit shearing stress with subscripts as shown in text (see also Table 7, Eq. 46c, and Fig. 14);
v = distances measured parallel to the v-axis;
W = total weight; load;
x = abscissa distance; coordinate with y and z;
y = ordinate (see under x);
z = coordinate (see under x);

α = slope of a reaction force (Fig. 6);

Δ = deformation; the deformation of the residual structure at the point of application of the redundants A, B, C, and D (identified by subscripts a, b, c, and d, respectively), and in the direction of the redundants A, B, C, and D, due to applied loads, volume changes, or yielding of foundations; subscripts L and T denote "longitudinal" and "transverse," respectively; subscripts R and T denote "rectangular" and "torsional," respectively;

δ = unit deformation; δ_{aa}, δ_{bc}, and δ_{ad} = the deformation at the point of application and in the direction of the redundant indicated by the first subscript, due to a unit load (or unit moment) acting at the point of application and in the direction of the redundant indicated by the second subscript;

ϵ = the tangent of the skew angle which is always positive;

θ = skew angle;

λ = modular ratio, $E_c/E_s = 2\,(1 + \mu)$; for concrete, $\lambda = 2.30$;

μ = Poisson's ratio; for concrete, $\mu = 0.15$; and

ϕ = the angle of the slope of a plane tangent to the neutral surface at point P.